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Dirichlet series expansions of *p*-adic L-functions

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Abstract

We study p-adic L-functions $L_p(s,\chi)$ for Dirichlet characters χ. We show that $L_p(s,\chi)$ has a Dirichlet series expansion for each regularization parameter c that is prime to p and the conductor of χ. The expansion is proved by transforming a known formula for p-adic L-functions and by controlling the limiting behavior. A finite number of Euler factors can be factored off in a natural manner from the p-adic Dirichlet series. We also provide an alternative proof of the expansion using p-adic measures and give an explicit formula for the values of the regularized Bernoulli distribution. The result is particularly simple for $c=2$, where we obtain a Dirichlet series expansion that is similar to the complex case.

Keywords p-adic L-Functions · Dirichlet Characters · Dirichlet Series · Euler Factors · Regularized Bernoulli Distributions · p-adic Measures

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1 Introduction

Let p be a prime, let $q=p$ if p is odd and $q=4$ if $p=2$, and let χ be a Dirichlet character of conductor f. A *p-adic L-function* $L_p(s,\chi)$ for a Dirichlet character χ is a p-adic meromorphic function and an analogue of the complex L-function. For powers of the Teichmüller character ω of conductor q, one obtains the *p-adic zeta functions* $\zeta_{p,i}=L_p(s,\omega^{1-i})$, where $i=0,1,\ldots,p-2$ ($i=0,1$ if $p=2$). It is well known that $L_p(s,\chi)$ is identically zero for odd χ. p-adic L-functions have a long history and the primary constructions going back to Kubota-Leopoldt [6] and Iwasawa [3] are via the interpolation of special values of complex L-functions.

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It can also be shown that p-adic L-functions are in fact Iwasawa functions.

It is well known that for $Re(s) > 0$,

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

and, more generally, if $c \geq 2$ is an integer,

$$(1 - \chi(c)c^{1-s})L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)\frac{a_{c,n}}{n^s},$$

where $a_{c,n} = 1 - c$ if $n \equiv 0 \bmod c$ and $a_{c,n} = 1$ if $n \not\equiv 0 \bmod c$. In the following, we derive similar, but slightly different, expansions for p-adic L-functions.

An explicit formula for $L_p(s, \chi)$ is given in [9] (Theorem 5.11): let F be any multiple of q and f. Then $L_p(s, \chi)$ is a meromorphic function (analytic if $\chi \neq 1$) on $\{s \in \mathbb{C}_p \mid |s| < qp^{-1/(p-1)}\}$ such that

$$L_p(s, \chi) = \frac{1}{F}\frac{1}{s-1} \sum_{\substack{a=1 \\ p \nmid a}}^{F} \chi(a)\langle a\rangle^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{j} \left(\frac{F}{a}\right)^j B_j. \tag{1}$$

In Sect. 2, we will use formula (1) to derive a Dirichlet series expansion of $L_p(s, \chi)$.

p-adic L-functions can be also be defined using distributions and measures. Let χ have conductor $f = dp^m$ with $(d, p) = 1$. Choose an integer $c \geq 2$, where $(c, dp) = 1$. Then there is a measure $E_{1,c}$ on $(\mathbb{Z}/d\mathbb{Z})^\times \times \mathbb{Z}_p^\times$ (the *regularized Bernoulli distribution*) such that

$$-(1 - \chi(c)\langle c\rangle^{1-s})L_p(s, \chi) = \int_{(\mathbb{Z}/d\mathbb{Z})^\times \times \mathbb{Z}_p^\times} \chi\omega^{-1}(a)\langle a\rangle^{-s}\, dE_{1,c} \tag{2}$$

(see [9] Theorem 12.2). In Sect. 3, we give an explicit formula for the values of $E_{1,c}$ and derive the Dirichlet series expansion from (2).

The expansion is particularly simple for $c = 2$, and this parameter can be used for $p \neq 2$ and Dirichlet characters with odd conductor. For this case we obtain similar results as in [1, 2], and [4]. In Sect. 4, we provide examples for different parameters c.

2 Expansions of *p*-adic *L*-functions

First, we derive an approximation of $L_p(s, \chi)$ that is close to the original definition of Kubota-Leopoldt (see [6]).

For $r \in \mathbb{C}_p^\times$ we write $\delta(r)$ for a term with p-adic absolute value $\leq |r|$.

Proposition 2.1 *Let p be a prime number, χ an even Dirichlet character of conductor f, and F a multiple of q and f. For $s \in \mathbb{C}_p$ with $|s| < qp^{-1/(p-1)}$, we have*

$$L_p(s,\chi) = \frac{1}{F}\frac{1}{s-1}\sum_{\substack{a=1\\ p\nmid a}}^{F} \chi(a)\langle a\rangle^{1-s} + \delta(F/qp). \tag{3}$$

Proof We use formula (1) above and look at the series $\sum_{j=0}^{\infty}\binom{1-s}{j}\left(\frac{F}{a}\right)^j B_j$. The first two terms are $1+(1-s)\frac{-F}{2a}$. We claim that the p-adic absolute value of the other terms ($j\geq 2$) is less than or equal to $|(s-1)F^2/qp|$. To this end, we note that $|1/j!|\leq p^{(j-1)/(p-1)}$ and

$$\left|\binom{1-s}{j}\right| \leq |1-s|\,p^{(j-1)/(p-1)}(qp^{-1/(p-1)})^{j-1} = |1-s|\,q^{j-1}$$

since we assumed that $|s| < qp^{-1/(p-1)}$. Since $|F|\leq \frac{1}{q}$, $|a|=1$, and $|B_j|\leq p$, we obtain

$$\left|\binom{1-s}{j}\left(\frac{F}{a}\right)^j B_j\right| \leq |1-s|\,q^{j-1}q^{2-j}|F|^2\,p = |1-s|\,|F|^2\,qp.$$

Then (1) implies

$$L_p(s,\chi) = \frac{1}{F}\frac{1}{s-1}\sum_{\substack{a=1\\ p\nmid a}}^{F}\chi(a)\langle a\rangle^{1-s} + \frac{1}{2}\sum_{\substack{a=1\\ p\nmid a}}^{F}\chi\omega^{-1}(a)\langle a\rangle^{-s} + \delta(F/qp).$$

It remains to show that the second sum can be absorbed into $\delta(F/qp)$. We have

$$\begin{aligned}\sum_{\substack{a=1\\ p\nmid a}}^{F}\chi\omega^{-1}(a)\langle a\rangle^{-s} &= \sum_{\substack{b=1\\ p\nmid b}}^{F}\chi\omega^{-1}(F-b)\langle F-b\rangle^{-s}\\ &= -\sum_{\substack{b=1\\ p\nmid b}}^{F}\chi\omega^{-1}(b)\langle b-F\rangle^{-s}\\ &= -\sum_{\substack{b=1\\ p\nmid b}}^{F}\chi\omega^{-1}(b)\langle b\rangle^{-s} + \delta(F/qp^{-1/(p-1)}).\end{aligned}$$

The last step can be justified by noting that

$$\frac{\langle b-F\rangle^{-s}}{\langle b\rangle^{-s}} = \left(1-\frac{F}{b}\right)^{-s} = 1+\sum_{j=1}^{\infty}\binom{-s}{j}\left(\frac{-F}{b}\right)^j = 1+\delta(F/qp^{-1/(p-1)}),$$

since $|s| < qp^{-1/(p-1)}$ (this is the same estimate as earlier, without the presence of the Bernoulli number). This proves the proposition. □

Remark 2.2 For $F = fp^n$ and $n\to\infty$, formula (3) gives the original definition of $L_p(s,\chi)$ by Kubota and Leopoldt (see [6]).

Remark 2.3 Suppose that $p\neq 2$. Then the error term in the above Proposition (as well as in the following Theorem 2.4) can be improved to $\delta(F/p^{2-(p-2)/(p-1)})$. First we note that $B_j = 0$ for odd $j\geq 3$. By the von Staudt–Clausen Theorem (see [9] 5.10), we have for even $j\geq 2$:

$|B_j| = p$ iff $(p-1) \mid j$, and otherwise $|B_j| \le 1$. Furthermore, $|1/j!| = p^{(j-S_j)/(p-1)}$, where S_j is the sum of the digits of j, written to the base p (see [5]). Since $j \equiv S_j \bmod (p-1)$, $j \equiv 0 \bmod (p-1)$ is equivalent to $S_j \equiv 0 \bmod (p-1)$. We conclude that $|B_j| = p$ yields $S_j \ge p-1$ and $|1/j!| \le p^{(j-1)/(p-1)} p^{-(p-2)/(p-1)}$. This implies the above error term. We also see that this error term cannot be further improved. ♢

Now we give the Dirichlet expansion of $L_p(s,\chi)$. For $m \in \mathbb{N}$, we denote by $\{x\}_m$ the unique representative of x mod $m\mathbb{Z}$ between 0 and $m-1$.

Theorem 2.4 *Let p be a prime number, χ be an even Dirichlet character of conductor f, and F a multiple of q and f. Let $c > 1$ be an integer satisfying $(c, F) = 1$. For $a \in \mathbb{Z}$, define*

$$\epsilon_{a,c,F} = \frac{c-1}{2} - \{-aF^{-1}\}_c \in \left\{ -\frac{c-1}{2}, -\frac{c-1}{2}+1, \dots, \frac{c-1}{2} \right\}.$$

Then we have for $s \in \mathbb{C}_p$ with $|s| < qp^{-1/(p-1)}$ the formula

$$-(1-\chi(c)\langle c\rangle^{1-s})L_p(s,\chi) = \sum_{\substack{a=1\\ p\nmid a}}^{F} \chi\omega^{-1}(a)\langle a\rangle^{-s}\epsilon_{a,c,F} + \delta(F/qp).$$

Proof Use (3) with cF in place of F, and subtract $\chi(c)\langle c\rangle^{1-s}$ times (3) with F, to obtain

$$\begin{aligned}(1-\chi(c)\langle c\rangle^{1-s})L_p(s,\chi) = {} & \frac{1}{cF}\frac{1}{s-1}\sum_{\substack{a=1\\ p\nmid a}}^{cF} \chi(a)\langle a\rangle^{1-s} \\ & - \frac{1}{F}\frac{1}{s-1}\sum_{\substack{a=1\\ p\nmid a}}^{F} \chi(ac)\langle ac\rangle^{1-s} + \delta(F/qp).\end{aligned} \tag{4}$$

Let $0 < a_0 < F$ with $(a_0, p) = 1$. Since we assumed $(c, F) = 1$ and $p \mid F$, there is a unique number of the form a_0c with $0 < a_0c < cF$ and $(a_0c, p) = 1$ in each congruence class modulo F relatively prime to p. The first sum in (4) can be written as

$$\begin{aligned}& \frac{1}{cF}\frac{1}{s-1}\sum_{\substack{a_0=1\\ p\nmid a_0}}^{F} \chi(a_0c)\langle a_0c\rangle^{1-s}\left(\sum_{\substack{a=1\\ a\equiv a_0c \bmod F}}^{cF} \left\langle 1+\frac{a-a_0c}{a_0c}\right\rangle^{1-s}\right) \\ = {} & \frac{1}{cF}\frac{1}{s-1}\sum_{\substack{a_0=1\\ p\nmid a_0}}^{F} \chi(a_0c)\langle a_0c\rangle^{1-s}\left(\sum_{\substack{a=1\\ a\equiv a_0c \bmod F}}^{cF} \left(1+(1-s)\frac{a-a_0c}{a_0c}\right)\right) + \delta(F/q).\end{aligned}$$

Note that $|\frac{a-a_0c}{a_0c}| \le |F|$, so this is the same type of estimate used in the proof of Proposition 2.1. Subtracting the second sum in (4) yields

$$
\begin{aligned}
&(1-\chi(c)\langle c\rangle^{1-s})L_p(s,\chi)\\
&=\frac{-1}{cF}\sum_{\substack{a_0=1\\ p\nmid a_0}}^{F}\chi(a_0c)\langle a_0c\rangle^{1-s}\left(\sum_{\substack{a=1\\ a\equiv a_0c \mod F}}^{cF}\frac{a-a_0c}{a_0c}\right)+\delta(F/qp)\\
&=\frac{-1}{c}\sum_{\substack{a_0=1\\ p\nmid a_0}}^{F}\chi\omega^{-1}(a_0c)\langle a_0c\rangle^{-s}\left(\sum_{\substack{a=1\\ a\equiv a_0c \mod F}}^{cF}\frac{a-a_0c}{F}\right)+\delta(F/qp).
\end{aligned}
$$

We compute the inner sum. Let $b=\{a_0c\}_F$. Then $a_0c=b+\{-F^{-1}b\}_c F$, since the latter sum is congruent to b modulo F and congruent to 0 modulo c. If a satisfies $a\equiv a_0c \mod F$ and $0<a<cF$, then $a=b+jF$ with $0\le j<c$. Hence

$$
\sum_{\substack{a=1\\ a\equiv a_0c \mod F}}^{cF}\frac{a-a_0c}{F}=\sum_{j=0}^{c-1}(j-\{-F^{-1}b\}_c)=c\,\epsilon_{b,c,F}\ .
$$

Since $b\equiv a_0c \mod F$, we have $\chi\omega^{-1}(b)\langle b\rangle^{-s}=\chi\omega^{-1}(a_0c)\langle a_0c\rangle^{-s}+\delta(F/q)$ by the same estimate as earlier, so

$$
-(1-\chi(c)\langle c\rangle^{1-s})L_p(s,\chi)=\sum_{\substack{b=1\\ p\nmid b}}^{F}\chi\omega^{-1}(b)\langle b\rangle^{-s}\epsilon_{b,c,F}+\delta(F/qp).
$$

This completes the proof. □

We can take the limit of $F=fp^n$ as $n\to\infty$ and obtain:

Corollary 2.5 *Let p be a prime number, χ an even Dirichlet character of conductor f, and $c>1$ an integer satisfying $(c,pf)=1$. Then we have for $s\in\mathbb{C}_p$ with $|s|<qp^{-1/(p-1)}$,*

$$
-(1-\chi(c)\langle c\rangle^{1-s})L_p(s,\chi)=\lim_{n\to\infty}\sum_{\substack{a=1\\ p\nmid a}}^{fp^n}\chi\omega^{-1}(a)\frac{\epsilon_{a,c,fp^n}}{\langle a\rangle^s}.
$$

The next Theorem shows that a finite number of Euler factors can be factored off in a similar way as in [8], where a *weak Euler product* was obtained. The main statement is that the remaining Dirichlet series has the expected form, similar to the complex case.

Theorem 2.6 *Let p be a prime number and let χ be an even Dirichlet character of conductor f. Let S be any finite (or empty) set of primes not containing p and set $S^+=S\cup\{p\}$. Let F be a multiple of q, f and all primes in S. Let $c>1$ be an integer satisfying $(c,F)=1$. Then we have for $s\in\mathbb{C}_p$ with $|s|<qp^{-1/(p-1)}$ the formula*

$$
-(1-\chi(c)\langle c\rangle^{1-s})\cdot\prod_{l\in S}(1-\chi\omega^{-1}(l)\langle l\rangle^{-s})\ \cdot\ L_p(s,\chi)=\sum_{\substack{a=1\\ (a,S^+)=1}}^{F}\chi\omega^{-1}(a)\frac{\epsilon_{a,c,F}}{\langle a\rangle^s}\ +\ \delta(F/qp).
$$

Proof We prove the statement by induction on $|S|$. By Theorem 2.4, the formula is true for $S = \emptyset$. Now assume the formula is true for S, and $l \neq p$ is a prime with $l \notin S$ and $(c, l) = 1$. It suffices to prove the following formula:

$$(1 - \chi\omega^{-1}(l)\langle l\rangle^{-s}) \sum_{\substack{a=1 \\ (a,S^+)=1}}^{F} \chi\omega^{-1}(a)\langle a\rangle^{-s}\epsilon_{a,c,F} = \sum_{\substack{a=1 \\ (a,S^+\cup\{l\})=1}}^{lF} \chi\omega^{-1}(a)\langle a\rangle^{-s}\epsilon_{a,c,lF} + \delta(F/qp). \tag{5}$$

Note that $|1 - \chi\omega^{-1}(l)\langle l\rangle^{-s}| \leq 1$ and $|lF| = |F|$, so we can keep the error term. We can use lF in place of F and write the left side of (5) as

$$\sum_{\substack{a=1 \\ (a,S^+)=1}}^{lF} \chi\omega^{-1}(a)\langle a\rangle^{-s}\epsilon_{a,c,lF} - \sum_{\substack{a=1 \\ (a,S^+)=1}}^{F} \chi\omega^{-1}(la)\langle la\rangle^{-s}\epsilon_{a,c,F} + \delta(F/qp). \tag{6}$$

Now we have

$$\epsilon_{la,c,lF} = \frac{c-1}{2} - \{-la(lF)^{-1}\}_c = \frac{c-1}{2} - \{-aF^{-1}\}_c = \epsilon_{a,c,F}.$$

Thus (6) is equal to

$$\begin{aligned} &\sum_{\substack{a=1 \\ (a,S^+)=1}}^{lF} \chi\omega^{-1}(a)\langle a\rangle^{-s}\epsilon_{a,c,lF} - \sum_{\substack{a=1 \\ (a,S^+)=1}}^{F} \chi\omega^{-1}(la)\langle la\rangle^{-s}\epsilon_{la,c,lF} + \delta(F/qp) \\ &= \sum_{\substack{a=1 \\ (a,S^+)=1 \\ l\nmid a}}^{lF} \chi\omega^{-1}(a)\langle a\rangle^{-s}\epsilon_{a,c,lF} + \delta(F/qp), \end{aligned}$$

which shows equation (5).

□

Remark 2.7 What happens if S contains more and more primes? It is well known that the Euler product does not converge p-adically (see [2]), since the factors $(1 - \chi\omega^{-1}(l)\langle l\rangle^{-s})$ have absolute value ≤ 1 and do not converge to 1 as $l \to \infty$. Furthermore, there are infinitely many primes l with $\chi\omega^{-1}(l) = 1$ and $(1 - \langle l\rangle^{-s})^{-1}$ has a pole at $s = 0$. We have for $l \neq p$ and $|s| < qp^{-1/(p-1)}$,

$$1 - \langle l\rangle^{-s} = -\sum_{j=1}^{\infty} \binom{-s}{j} (\langle l\rangle - 1)^j.$$

The p-adic absolute value of each term of the above series is less than

$$(qp^{-1/(p-1)})^j p^{(j-1)/(p-1)} q^{-j} = p^{-1/(p-1)} < 1.$$

Hence the product $\prod_{l\in S}(1 - \chi\omega^{-1}(l)\langle l\rangle^{-s})$ approaches 0 as S expands to include all primes.

3 Regularized Bernoulli distributions

Let p be a prime number and let d be a positive integer with $(d,p)=1$. Define $X_n = (\mathbb{Z}/dp^n\mathbb{Z})$ and $X = \varprojlim X_n \cong \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}_p$. Let $k \geq 1$ be an integer. Then the *Bernoulli distribution* E_k on X is defined by

$$E_k(a + dp^nX) = (dp^n)^{k-1}\frac{1}{k}B_k\left(\frac{\{a\}_{dp^n}}{dp^n}\right),$$

where $B_k(x)$ is the k-th Bernoulli polynomial and $B_k = B_k(0)$ are the Bernoulli numbers (see [5, 7]). For $k=1$, one has $B_1(x) = x - \frac{1}{2}$. Choose $c \in \mathbb{Z}$ with $c \neq 1$ and $(c, dp) = 1$. Then the *regularization* $E_{k,c}$ of E_k is defined by

$$E_{k,c}(a + dp^nX) = E_k(a + dp^nX) - c^kE_k\left(\left\{\frac{a}{c}\right\}_{dp^n} + dp^nX\right).$$

One shows that the regularized Bernoulli distributions $E_{k,c}$ are measures (see [7]). In the following, we consider only $k=1$; the cases $k \geq 2$ are similar.

Theorem 3.1 *Let p be a prime, c, $d \in \mathbb{N}$, and $c \geq 2$ such that $(c, dp) = 1$. Let X be as above, and let $E_{1,c}$ be the regularized Bernoulli distribution on X. For $a \in \{0, 1, \dots, dp^n - 1\}$, we have*

$$E_{1,c}(a + dp^nX) = \frac{c-1}{2} - \{-a(dp^n)^{-1}\}_c = \epsilon_{a,c,dp^n}.$$

Proof By definition,

$$E_{1,c}(a + dp^nX) = E_1(a + dp^nX) - cE_1(c^{-1}a + dp^nX) = \frac{a}{dp^n} - \frac{1}{2} - c\left(\frac{\{c^{-1}a\}_{dp^n}}{dp^n}\right) + \frac{c}{2}.$$

We give the standard representative of $c^{-1}a \bmod dp^n$:

$$\{c^{-1}a\}_{dp^n} = \frac{\{-a(dp^n)^{-1}\}_c\, dp^n + a}{c}$$

Note that the numerator is divisible by c, since $\{-a(dp^n)^{-1}\}_c\, dp^n \equiv -a \bmod c$. Hence the quotient is an integer between 0 and $dp^n - 1$. Furthermore, the numerator is congruent to a modulo dp^n, and so the quotient has the desired property. We obtain

$$E_{1,c}(a + dp^nX) = \frac{a}{dp^n} + \frac{c-1}{2} - \frac{\{-a(dp^n)^{-1}\}_c\, dp^n + a}{dp^n} = \frac{c-1}{2} - \{-a(dp^n)^{-1}\}_c$$

which is the assertion. □

Now the Dirichlet series expansion in Corollary 2.5 follows from Theorem 3.1 and the integral formula (2).

4 Expansions for different regularization parameters

We look at the coefficients ϵ_{a,c,dp^n} for different parameters c and the resulting Dirichlet series expansions. The following observation follows directly from the definition.

Remark 4.1 The sequence of values $E_{1,c}(a + dp^n X) = \epsilon_{a,c,dp^n}$ for $a = 0,1, 2, \cdots, dp^n - 1$ is periodic with period c. The sequence begins with $\frac{c-1}{2}$ and continues with a permutation of $\frac{c-3}{2}, \ldots, -\frac{c-1}{2}$. If we restrict to values of n such that dp^n lies in a fixed congruence class modulo c, then the values do not change as $n \to \infty$. ♢

The measure $E_{1,c}$ and the Dirichlet series expansion are particularly simple for $c = 2$. Note that we assumed that d and p are odd in this case. If a is even, then $\{-a(dp^n)^{-1}\}_2 = 0$ and

$$E_{1,2}(a + dp^n X) = \epsilon_{a,2,dp^n} = \frac{1}{2}.$$

If a is odd, then $-a(dp^n)^{-1}$ is odd, $\{-a(dp^n)^{-1}\}_2 = 1$ and

$$E_{1,2}(a + dp^n X) = \epsilon_{a,2,dp^n} = -\frac{1}{2}.$$

Hence $E_{1,2}$ is up to the factor $\frac{1}{2}$ equal to the following simple measure:

Definition 4.2 Let $p \neq 2$ be a prime, and let $X \cong \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}_p$ be as above. Then

$$\mu(a + dp^n X) = (-1)^{\{a\}_{dp^n}}$$

defines a measure on X. We call μ the *alternating measure*, since the measure of all clopen balls is ± 1. ♢

The corresponding integral is also called the *fermionic* p-adic integral (see [4]).
Now we obtain the following Dirichlet series expansion from Corollary 2.5.

Corollary 4.3 *Let $p \neq 2$ be a prime number, and let χ be an even Dirichlet character of odd conductor f. Then we have for $s \in \mathbb{C}_p$ with $|s| < p^{(p-2)/(p-1)}$,*

$$(1 - \chi(2)\langle 2\rangle^{1-s})L_p(s,\chi) = \lim_{n\to\infty} \frac{1}{2} \sum_{\substack{a=1 \\ p\nmid a}}^{fp^n} (-1)^{a+1} \chi\omega^{-1}(a) \frac{1}{\langle a\rangle^s}.$$

For $\chi = \omega^{1-i}$ and odd $i = 1, \ldots, p-2$, we obtain the branches of the p-adic zeta function:

$$\zeta_{p,i}(s) = L_p(s, \omega^{1-i}) = \frac{1}{1 - \omega(2)^{1-i}\langle 2\rangle^{1-s}} \cdot \lim_{n\to\infty} \frac{1}{2} \sum_{\substack{a=1 \\ p\nmid a}}^{p^n} (-1)^{a+1} \omega(a)^{-i} \frac{1}{\langle a\rangle^s}$$

Remark 4.4 Dirichlet series expansions of p-adic L-functions were studied by D. Delbourgo in [1] and [2]. He considers Dirichlet characters χ satisfying $(p, 2f\phi(f)) = 1$ and their Teichmüller twists. We obtain the same expansion for $c = 2$ and $\chi = \omega^{1-i}$. However, we require $(c, fp) = 1$ and use other methods for the proof.

Similar expansions for a slightly different p-adic L-function using a fermionic p-adic integral (i.e., $c = 2$) were also obtained by M.-S. Kim and S. Hu (see [4]).

Example 4.5 We look at the case $c = 3$. The sequence of values $\epsilon_{a,3,dp^n}$ is periodic with period 3. If $dp^n \equiv 1 \bmod 3$, then the sequence is $1, -1, 0, \dots$. If $dp^n \equiv 2 \bmod 3$, then we obtain the sequence $1, 0, -1, \dots$.

Corollary 4.6 *Let p be a prime number, and let χ be an even Dirichlet character of conductor $f = dp^m$ such that $(3, dp) = 1$. If $d \equiv 1 \bmod 3$, then define a sequence $\epsilon_0 = 1, \epsilon_1 = -1, \epsilon_2 = 0, \cdots$ with period 3. Otherwise, set $\epsilon_0 = 1, \epsilon_1 = 0, \epsilon_2 = -1$ and extend it with period 3. Then we have for $s \in \mathbb{C}_p$ with $|s| < qp^{-1/(p-1)}$,*

$$-(1 - \chi(3)\langle 3\rangle^{1-s})L_p(s, \chi) = \lim_{n\to\infty} \sum_{\substack{a=1 \\ p\nmid a}}^{dp^{2n}} \chi\omega^{-1}(a)\frac{\epsilon_a}{\langle a\rangle^s}.$$

Example 4.7 For $c = 5$, we get a periodic sequence with period 5 and we have $\epsilon_{a,5,dp^n} = 2$ for $a \equiv 0 \bmod 5$. The next four coefficients are a permutation of the values $-2, -1, 0$ and 1, depending on the class of $dp^n \bmod 5$.

Example 4.8 Let $c = 7$. Then $\epsilon_{0,7,dp^n} = 3$. Now suppose, for example, that $dp^n \equiv 3 \bmod 7$. Then $(dp^n)^{-1} \equiv 5 \bmod 7$. This yields the values

$$\epsilon_{1,7,dp^n} = 1,\ \epsilon_{2,7,dp^n} = -1,\ \epsilon_{3,7,dp^n} = -3,\ \epsilon_{4,7,dp^n} = 2,\ \epsilon_{5,7,dp^n} = 0,\ \epsilon_{6,7,dp^n} = -2,$$

and these are extended with period 7.

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